$$\int_0^{19}\int_x^{\sqrt{38x-x^2}} 9\sqrt{x^2+y^2}\,dydx =$$

Solution:

The integral limits can be expressed as

$0 \leq x \leq 19$ and $x \leq y \leq \sqrt{38x-x^2}$

It is always a good practice to draw the graph of the limits

$y = x$ is a straight line.

$y = \sqrt{38x-x^2}$ is the equation of a circle how ??

Square both sides:

$$y^2 = 38x - x^2 \Rightarrow$$

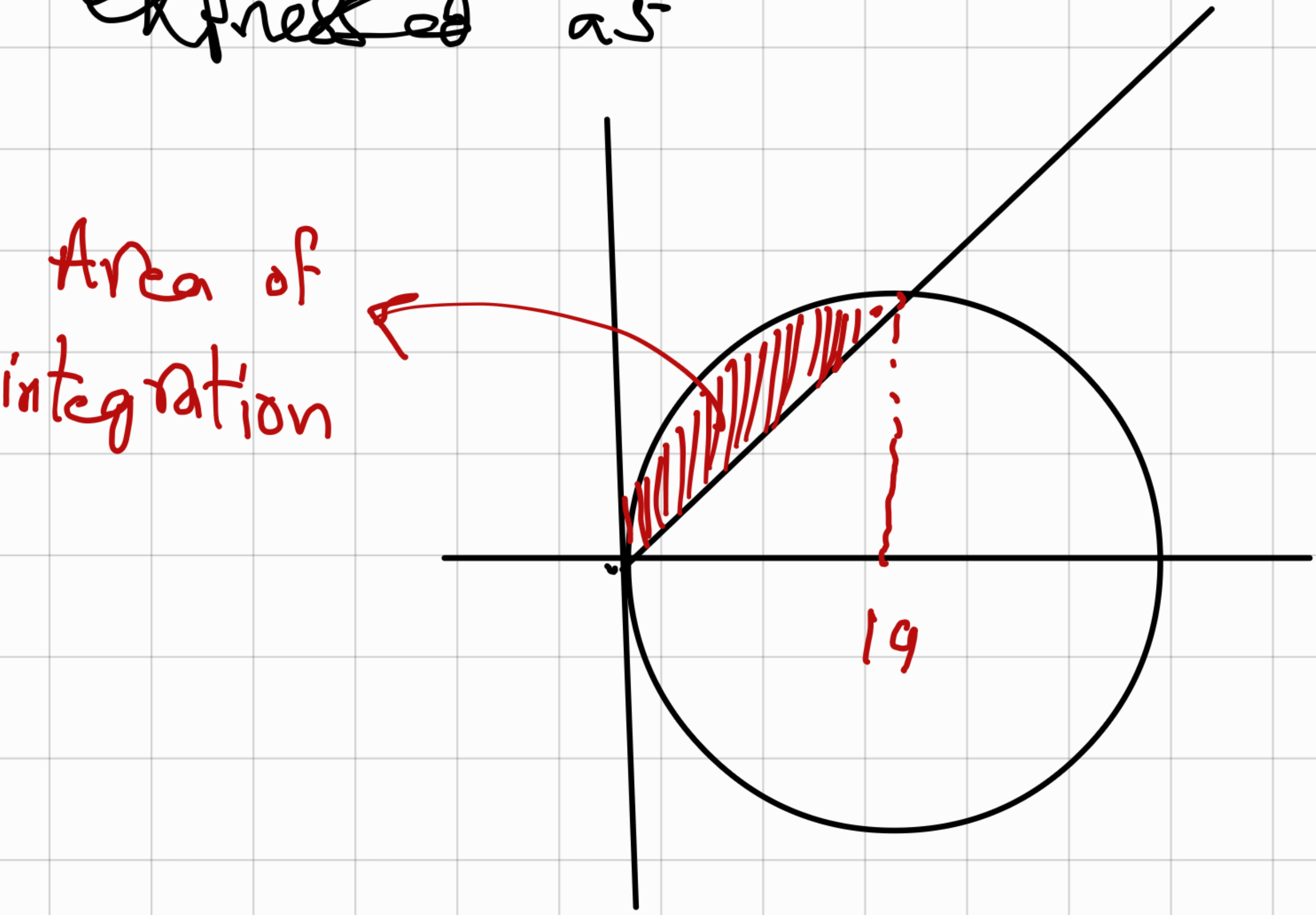
$$x^2 - 38x + y^2 = 0$$

$$+(19)^2 \qquad\qquad +(19)^2$$

$$\left(x^2 - 38x + (19)^2\right) + y^2 = (19)^2$$

$$\underbrace{(x - 19)^2 + y^2 = (19)^2}_{\text{equation of a circle of radius 19 and centered at } (19,0)}$$

The limits can be grafically expressed as



<u>Remember:</u> The difficulty here in solving the double integral lies in the integration limits

By noticing that the function

$y = \sqrt{38x - x^2}$ can be

written as

$$y^2 = 38x - x^2 \quad \text{(Squaring both sides of the equat)}$$

$$x^2 + y^2 = 38x$$

Using Polar coordinates we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad x^2 + y^2 = r^2$$

Therefore,

$$\underbrace{x^2 + y^2}_{= r^2} = \underbrace{38x}_{= r\cos\theta}$$

$$r^2 = 38\, r \cos\theta$$

$$r^2 - 38\, r\cos\theta = 0$$

$$r\,(r - 38\cos\theta) = 0$$

either $r = 0$ or $r = 38\cos\theta$

Then the limits become

$$0 \leq r \leq 38\cos\theta$$

and $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

The above integral becomes

$$I = \int_{\pi}^{\pi/2} \int_{0}^{38\cos\theta} 9 \underbrace{\sqrt{r^2}}_{=|r| = r \text{ bec}} \, r\, dr\, d\theta$$

$\frac{\pi}{4}$

limits are positive

$$= 9 \int_{\pi/4}^{\pi/2} \int_{0}^{38\cos\theta} 9 r^2 \, dr \, d\theta$$

$$= 9 \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{38\cos\theta} d\theta$$

$$= \frac{9}{3} \int_{\pi/4}^{\pi/2} (38\cos\theta)^3 \, d\theta$$

$$= 3 \times (38)^3 \cdot \int_{\pi/4}^{\pi/2} \cos^3\theta \, d\theta$$

$\pi/2$

$$= 3(38)^3 \cdot \int_{\pi/4} \cos\theta \, \underbrace{\cos^2\theta}_{(1-\sin^2\theta)} \, d\theta$$

$$= 3(38)^3 \left[\int_{\pi/4}^{\pi/2} \cos\theta \, d\theta - \int_{\pi/4}^{\pi/2} \cos\theta \sin^2\theta \, d\theta \right]$$

$$= 3(38)^3 \left[\sin\theta \Big|_{\pi/4}^{\pi/2} - \frac{\sin^3\theta}{3} \Big|_{\pi/4}^{\pi/2} \right]$$

$$= 3(38)^3 \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4} - \frac{\sin^3\left(\frac{\pi}{2}\right) - \sin^3\left(\frac{\pi}{4}\right)}{3} \right]$$

$$\simeq 3(38)^3 \times 0.077411\ldots$$

$$\simeq \boxed{12743.1}$$

$$0.0774110157 \times 3 \times 38^{(3)}$$

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