2 Suppose you choose a real number X from the interval $[2, 10]$ with a density function of the form

$$f(x) = Cx \ ,$$

where C is a constant.

(a) Find C.

(b) Find $P(E)$, where $E = [a, b]$ is a subinterval of $[2, 10]$.

(c) Find $P(X > 5)$, $P(X < 7)$, and $P(X^2 - 12X + 35 > 0)$.

Solution:

(a) The density function of x is only defined on the interval $[2, 10]$, therefore we can rewrite $f_X(x)$ as:

$$f_X(x) = \begin{cases} Cx & 2 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Rmember

The condition on $f_X(x)$ to be a valid probability density function

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

and that $f_X(x) > 0$

Therefore, we have:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_2^{10} Cx\,dx = 1 \Rightarrow$$

$$= C\int_2^{10} x\,dx = 1 \Rightarrow$$

$$= C\left[\frac{x^2}{2}\right]_2^{10} = 1 \Rightarrow$$

$$= \frac{C}{2}\left[10^2 - 2^2\right] = 1 \Rightarrow$$

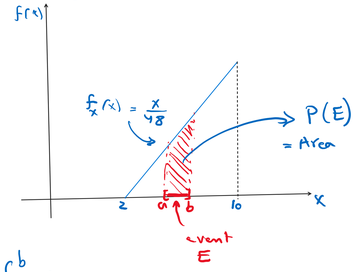
$$= \frac{C}{2}\left[100 - 4\right] = 1 \Rightarrow$$

$$= C \times \frac{96}{2} = 1 \Rightarrow$$

$$= C = \frac{2}{96} \Rightarrow$$

$$\boxed{C = \frac{1}{48}}$$

(b) The event E can be expressed graphically as



$$P(E) = \int_a^b f_X(x)\,dx$$

$$= \int_a^b \frac{x}{48}\,dx = \frac{1}{48}\int_a^b x\,dx$$

$$= \frac{1}{48}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{48 \times 2}\left[x^2\right]_a^b$$

$$= \frac{b^2 - a^2}{96} \quad \left[\begin{array}{l}\text{for } a \geq 2 \\ \text{and } b \leq 10\end{array}\right]$$

(c) $P(X > 5) = P(\{5 < X \leq 10\} \cup \{X > 10\})$

$$= P(5 < X \leq 10) + \underbrace{\cancel{P(X > 10)}}_{=0}$$

Same as $P(E)$ $a = 5,\ b = 10$

Since $f_X(x) = 0$ for $x > 10$

$$= \frac{10^2 - 5^2}{96}$$

$$= \frac{100 - 25}{96} = \frac{75}{96} = \frac{25}{32}$$

$P(X < 7) = P(\{X < 2\} \cup \{2 \leq X < 7\})$

$$= \underbrace{\cancel{P(X < 2)}}_{=0} + P(2 \leq X < 7)$$

Since $f_X(x) = 0$ for $x < 2$

$= P(E)$ $a = 2,\ b = 7$

$$= \frac{7^2 - 2^2}{96} = \frac{49 - 4}{96} = \frac{45}{96}$$

$$= \frac{15}{32}$$

$P(X^2 - 12X + 35 > 0)$

Consider the following inequality

$$X^2 - 12X + 35 > 0$$

$$\underbrace{(X - 7)}_{a}\underbrace{(X - 5)}_{b} > 0$$

Using factorization

The product of two terms a and b is positive if and only if

$\boxed{1}$ $a > 0$ and $b > 0$

or

$\boxed{2}$ $a < 0$ and $b < 0$

Case $\boxed{1}$: this event can be represented by:

$X - 7 > 0$ and $X - 5 > 0$

$\{X > 7\} \cap \{X > 5\}$

$= \{X > 7\}$

Case $\boxed{2}$: this event can be represented by:

$X - 7 < 0$ and $X - 5 < 0$

$\{X < 7\} \cap \{X < 5\}$

$= \{X < 5\}$

Therefore,

$P(X^2 - 12X + 35 > 0)$

$= P(\{X > 7\} \cup \{X < 5\})$ OR

Two disjoint events

$= P(X > 7) + P(X < 5)$

$= 1 - P(X \leq 7) + 1 - P(X \geq 5)$

$= 2 - [P(X \leq 7) + P(X \geq 5)]$

$= 2 - \left(\frac{45}{96} + \frac{75}{96}\right) = \frac{192 - 120}{96}$

$= \frac{72}{96} = \boxed{\frac{1}{4}}$